MATH 5485 Introduction to Numerical Methods I

Instructor: *Alexander Shapeev* September 13, 2013

Selected Answers to Quiz 1

Problem 1 (20pt). For a sequence $x_n = n\sin(1/n)$, compute the limit and determine the rate of convergence. (Hint: use the Taylor's expansion of $\sin x$ for $x \approx 0$.)

Solution. Expand $\sin(x) = x + O(x^3)$ (near $x = 0$), hence as $n \to \infty$, $x_n = n\sin(1/n) = n(1/n + O(1/n^3)) = 1 + O(1/n^2)$. Thus, the limit is 1 and the rate is $1/n^2$.

Problem 2 (10pt). For the following sequence,

n	p_n
0	7.10000
1	7.06641
2	7.03536
3	7.01354
4	7.00317
5	7.00036

find out numerically whether its order of convergence to 7 equals $\alpha = 1.5$. If yes then determine the asymptotic error constant λ.

Solution. Compute the quantities $|p_n - 7|/|p_{n-1} - 7|^{1.5}$ for $n = 1, 2, 3, 4, 5$ and observe that they approach 2. This confirms numerically that the order of convergence is 1.5 and $\lambda \approx 2$.

Problem 3 (10pt). Sec. 1.3, Exercise 1(c).

Problem 4 (20pt). Sec. 1.3, Exercise 8.

Problem 5 (20pt). Sec. 1.3, Exercise 16.

Solution. The solution to the equation with the initial condition x_0 is

$$x(t) = \frac{\pi}{2}\frac{x_0}{t} - \frac{\cos(t)}{t}.$$

The solution with the initial condition $x_0 + \epsilon$ is

$$y(t) = \frac{\pi}{2}\frac{x_0 + \epsilon}{t} - \frac{\cos(t)}{t}.$$

The difference is

$$y(t) - x(t) = \frac{\epsilon}{t}$$

and it decreases as $t \to +\infty$. Hence the problem is *well-conditioned.*

Problem 6 (20pt). Sec. 1.4, Exercise 10.

Solution. (b) $y(x) \approx -\frac{x^3}{6}$ (or, e.g., $-\frac{x^3}{6} + \frac{23x^5}{120}$)