MATH 5485 Introduction to Numerical Methods I

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Answers to Quiz 7 (take-home)

Problem 1 (20pt). Consider an $n \times n$ tridiagonal matrix A with all diagonal elements being 2 and all off-diagonal elements being -1:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let b be the following n-dimensional vector

$$b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

For each of the following values of n, solve the system $Ax = b$ using the conjugate gradients method. Take $x^{(0)} = 0$ and terminate iterations when the norm of the residual, $\|r^{(k)}\|_2$, falls below 10^{-6}. Report the ℓ_∞-norm of the solution, $\|x\|_\infty$, and the number of iterations k.

(a) $n = 32$
Sol. $\|x\|_\infty = 136$, $k = 16$

(b) $n = 64$
Sol. $\|x\|_\infty = 528$, $k = 32$

(c) $n = 128$
Sol. $\|x\|_\infty = 2080$, $k = 64$

Additionally:

(d) Give the details (here or on a separate paper) of how you obtained the answer (formulae, codes, Matlab commands, etc.). If you used the code from the book's author, you do not need to provide it here.

(e) How does the number of iterations k grow compared to n (hint: evaluate the ratio k/n).
Sol. $k/n = 1/2$, so $n = k/2$.

Problem 2 (20pt). Let $\tilde{A} = (A^T A)(A^T A)$ and $\tilde{b} = (A^T A)(A^T b)$, where A and b are defined in Problem 1.

(a) How does the solution of the equation $\tilde{A}x = \tilde{b}$ compare to the solution of the equation $Ax = b$ (from Problem 1)?
Sol. The solutions are the same because the equations $Ax = b$ and $MAx = Mb$ have the same solutions if M is an invertible matrix.

For each of the following values of n, solve the system $\tilde{A}x = \tilde{b}$ using the conjugate gradients method. Take $x^{(0)} = 0$ and terminate iterations when the norm of the residual, $\|r^{(k)}\|_2$, falls below 10^{-6}. Report the ℓ_∞-norm of the solution, $\|x\|_\infty$, and the number of iterations k.

(b) $n = 32$
Sol. $\|x\|_\infty = 136$, $k \approx 100$

(c) $n = 64$
Sol. $20 \lessapprox \|x\|_\infty \lessapprox 80$, $k \approx 250$

(d) $n = 128$
Sol. $20 \lessapprox \|x\|_\infty \lessapprox 80$, $k \approx 2500$

Additionally:

(e) Give the details (here or on a separate paper) of how you obtained the answer (formulae, codes, Matlab commands, etc.). If you used the code from the book's author, you do not need to provide it here.

(f) Does the number of iterations k grow faster or slower than n (hint: evaluate the ratio k/n)?
Sol. k/n increases, so k grows faster than n.

Problem 3 (20pt). Consider using Newton method for finding eigenvalues and eigenvectors of a matrix

$$A = \begin{pmatrix} 5 & 1 \\ 1 & -6 \end{pmatrix}.$$

(Notice that A is a symmetric matrix.) A vector $v = (v_1, v_2)^T$ and a number λ is an eigenvector-eigenvalue pair of A if $\lambda v = Av$, or in other words,

$$\begin{aligned} \lambda v_1 &= 5v_1 + v_2 \\ \lambda v_2 &= 1v_1 - 6v_2 \end{aligned}$$

We additionally require that $\frac{1}{2}\|v\|_2^2 \equiv \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 = \frac{1}{2}$. Introducing $x = (x_1, x_2, x_3)^T$ such that $x_1 = v_1$, $x_2 = v_2$ and $x_3 = \lambda$, we can rewrite the system as $F(x) = 0$, where

$$F(x) = \begin{pmatrix} x_3x_1 - 5x_1 - x_2 \\ x_3x_2 - x_1 + 6x_2 \\ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{2} \end{pmatrix}.$$

(a) Write down $J(x) = \nabla F(x)$. (Hint: your $J(x)$ should be a symmetric matrix for all x.)

The (large) elements on the diagonal of A, namely 5 and -6, should give a good approximation to the eigenvalues, and a corresponding approximation to the eigenvectors are $(1, 0)^T$ and $(0, 1)^T$.

(b) Starting with the initial guess $x_1 = 1$, $x_2 = 0$, $x_3 = 5$, apply the Newton method to $F(x) = 0$. Report the number of iterations and the solution $x = (x_1, x_2, x_3)^T$.
Sol. iterations: $k = 4$ for a stopping condition $\|x^{(k)} - x^{(k-1)}\| \leq 10^{-6}$ or $k = 3$ for a stopping condition $\|F(x^{(k)})\| \leq 10^{-6}$.

(c) Starting with the initial guess $x_1 = 0$, $x_2 = 1$, $x_3 = -6$, apply the Newton method to $F(x) = 0$. Report the number of iterations and the solution $x = (x_1, x_2, x_3)^T$.
Sol. iterations: $k = 4$ for a stopping condition $\|x^{(k)} - x^{(k-1)}\| \leq 10^{-6}$ or $k = 3$ for a stopping condition $\|F(x^{(k)})\| \leq 10^{-6}$.

(d) Compare your solutions to (b) and (c) with the eigenvalues and the eigenvectors of A.
Sol. They must coincide up to at least 5 digits (but you actually need to check it). (The eigenvectors may coincide up to change in the sign.)