MATH 5485 Introduction to Numerical Methods I

Instructor: *Alexander Shapeev* November 1, 2013

Quiz 7 (take-home)

due November 8, 2013

First Name(s):

Last Name:

Remember:

- Print this document out and give your answers here, except if you need to attach extra pages for lengthy details
- Work on your own.
- Justify your answers (especially when the answer is "yes" or "no", or a single number).
- Provide details (e.g., how to derive a solution).
- Do NOT use red color for your answers.
- Write legibly, especially the answers (if hand-written).

Problem 1 (20pt). Consider an $n \times n$ tridiagonal matrix A with all diagonal elements being 2 and all off-diagonal elements being -1:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let b be the following n-dimensional vector

$$b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

For each of the following values of n, solve the system $Ax = b$ using the conjugate gradients method. Take $x^{(0)} = 0$ and terminate iterations when the norm of the residual, $\|r^{(k)}\|_2$, falls below 10^{-6}. Report the $\boldsymbol{\ell_\infty}$**-norm of the solution, $\|\boldsymbol{x}\|_\infty$,** and the number of iterations k.

(a) $n = 32$

(b) $n = 64$

(c) $n = 128$

Additionally:

(d) Give the details (here or on a separate paper) of how you obtained the answer (formulae, codes, Matlab commands, etc.). If you used the code from the book's author, you do not need to provide it here.

(e) How does the number of iterations k grow compared to n (hint: evaluate the ratio k/n).

Problem 2 (20pt). Let $\tilde{A} = (A^T A)(A^T A)$ and $\tilde{b} = (A^T A)(A^T b)$, where A and b are defined in Problem 1.

(a) How does the solution of the equation $\tilde{A}x = \tilde{b}$ compare to the solution of the equation $Ax = b$ (from Problem 1)?

For each of the following values of n, solve the system $\tilde{A}x = \tilde{b}$ using the conjugate gradients method. Take $x^{(0)} = 0$ and terminate iterations when the norm of the residual, $\|r^{(k)}\|_2$, falls below 10^{-6}. Report the **ℓ_∞-norm of the solution, $\|x\|_\infty$,** and the number of iterations k.

(b) $n = 32$

(c) $n = 64$

(d) $n = 128$

Additionally:

(e) Give the details (here or on a separate paper) of how you obtained the answer (formulae, codes, Matlab commands, etc.). If you used the code from the book's author, you do not need to provide it here.

(f) Does the number of iterations k grow faster or slower than n (hint: evaluate the ratio k/n)?

Problem 3 (20pt). Consider using Newton method for finding eigenvalues and eigenvectors of a matrix

$$A = \begin{pmatrix} 5 & 1 \\ 1 & -6 \end{pmatrix}.$$

(Notice that A is a symmetric matrix.) A vector $v = (v_1, v_2)^T$ and a number λ is an eigenvector-eigenvalue pair of A if $\lambda v = Av$, or in other words,

$$\begin{aligned} \lambda v_1 &= 5v_1 + v_2 \\ \lambda v_2 &= 1v_1 - 6v_2 \end{aligned}$$

We additionally require that $\frac{1}{2}\|v\|_2^2 \equiv \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 = \frac{1}{2}$. Introducing $x = (x_1, x_2, x_3)^T$ such that $x_1 = v_1$, $x_2 = v_2$ and $x_3 = \lambda$, we can rewrite the system as $F(x) = 0$, where

$$F(x) = \begin{pmatrix} x_3x_1 - 5x_1 - x_2 \\ x_3x_2 - x_1 + 6x_2 \\ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{2} \end{pmatrix}.$$

(a) Write down $J(x) = \nabla F(x)$. (Hint: your $J(x)$ should be a symmetric matrix for all x.)

The (large) elements on the diagonal of A, namely 5 and -6, should give a good approximation to the eigenvalues, and a corresponding approximation to the eigenvectors are $(1, 0)^T$ and $(0, 1)^T$.

(b) Starting with the initial guess $x_1 = 1$, $x_2 = 0$, $x_3 = 5$, **and using the tolerance of 10^{-6}**, apply the Newton method to $F(x) = 0$. Report the number of iterations and the solution $x = (x_1, x_2, x_3)^T$.

(c) Starting with the initial guess $x_1 = 0$, $x_2 = 1$, $x_3 = -6$, **and using the tolerance of 10^{-6}**, apply the Newton method to $F(x) = 0$. Report the number of iterations and the solution $x = (x_1, x_2, x_3)^T$.

(d) Compare your solutions to (b) and (c) with the eigenvalues and the eigenvectors of A.