MATH 5485 Introduction to Numerical Methods I

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Quiz 8

First Name(s):

Last Name: Solutions

Rules:

- **There are 2 problems in this test.**
- **Answer all questions in each problem.**
- **Justify your answers** (especially when the answer is "yes" or "no", or a single number).
- **Provide details** (e.g., how to derive a solution).
- No unauthorized electronic devices (phones, computers, etc.)
- No written materials (books, notes, etc.)
- You can use your own draft paper or ask the instructor to provide some.
- Do NOT use red color for your answers.
- Write legibly, especially the answers.

Problem 1 (40pt). Consider the Gauss-Seidel method for the equation $Ax = b$ with

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}.$$

(a) Write down the iteration matrix T. (20pt)

Hint: use the following formula for matrix inversion: $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$.

(b) Find the spectral radius of T, $\rho(T)$, and hence comment on the convergence of the method. (20pt)

(a) Split $A = D - L - U$, where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad L = \begin{pmatrix} 0 & 0 \\ +1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & +\frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

$$T_{g.s.} \overset{def}{=} (D-L)^{-1}U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

used the hint

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}}.$$

(b) $\lambda_{1,2}(T) = 0, \frac{1}{2},$

$$\rho(T) = \max_i |\lambda_i| = \boxed{\frac{1}{2}.}$$

$\boxed{\rho(T) < 1 \Rightarrow \text{method conv}}$

Problem 2 (60pt). Consider the Newton method applied to the equation $f(x) = 0$ with $f(x) = \sin^2 x$.

(a) Find analytically the root on the interval $[3, 4]$. (20pt)

(b) Does the method converge if started sufficiently close to the root? (20pt)

(c) What is the order of convergence of the method? (20pt)

(a) root: $f(x)=0 \iff \sin^2 x = 0 \iff \sin x = 0$

$\iff \boxed{x=\pi} \leftarrow$ the only root on $[3,4]$

(they don't need to justify that $x=\pi$ is the only root of $\sin x$).

(b) Yes - the Newton's method always converges if ~~stra~~ started sufficiently close to the root

Alt: Newton's method;

$x^{(n+1)} = g(x^{(n)})$, where

$$g = x - \frac{f(x)}{f'(x)} = x - \frac{\sin^2 x}{2\sin x \cos x} = x - \frac{\tan x}{2}$$

$$g'(\pi) = 1 - \frac{1}{2\cos^2\pi} = 1 - \frac{1}{2} = \frac{1}{2}, \quad |g'(\pi)| < 1 \Rightarrow$$

hence converges

Alt: take an interval $[a,b] \ni \pi$

e.g. $[3, 3.2]$, and prove that

g maps $[3, 3.2]$ into $[3, 3.2]$ and

$|g'(x)| \leq k$ where $k<1$ everywhere on the interval

(c) π is at least a double root, since

$f(\pi)=0$ and $f'(\pi) = 2\sin\pi\cos\pi = 0$,

~~Hence~~ Hence $\boxed{\alpha \text{ (order of conv.)} = 1}$

Alt: $g'(\pi) = \frac{1}{2} \neq 0 \Rightarrow$ order of conv $= 1$